

Fermion–Antifermion Condensate Contribution to the Anomalous Magnetic Moment of a Fundamental Dirac Fermion

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We consider the contribution of fermion–antifermion condensates to the anomalous magnetic moment of a fermion in a vacuum in which such condensates exist. The real part of the condensate contribution to the anomalous magnetic moment is shown to be zero. A nonzero imaginary part is obtained below the kinematic threshold for intermediate fermion–antifermion pairs. The calculation is shown to be gauge-parameter independent provided a single fermion mass characterizes both the fermion propagator and condensate-sensitive contributions, suggestive of a dynamically generated fermion mass. The nonzero imaginary part is then argued to correspond to the kinematic production of the intermediate-state Goldstone bosons anticipated from a chiral-noninvariant vacuum. Finally, speculations are presented concerning the applicability of these results to quark electromagnetic properties.

1. INTRODUCTION

One of the key distinctions between quantum electrodynamics (QED) and quantum chromodynamics (QCD), field theories of known-interaction physics with unbroken gauge symmetry, is the existence of QCD-vacuum condensates, a distinction that has not been adequately linked to the nonabelian character of the latter theory. In particular, the quark–antiquark condensate $\langle \bar{q}q \rangle$ characterizes the chiral noninvariance of the QCD vacuum; QED has no corresponding electron–positron condensate. Although a dynamical breakdown in chiral invariance can be linked to a criticality threshold in the size of the gauge coupling constant (Higashijima, 1984), such an argument might

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in and of itself suggest the possible existence of an electron–positron condensate in the electromagnetic potential of atomic nuclei with sufficiently large atomic number. In such a scenario, the only linkage between the nonabelian character of QCD and the existence of vacuum condensates would be the large size of the running QCD coupling constant anticipated in the low-momentum (static-quark) limit.

It is therefore of interest to explore whether the existence of such a condensate might alter well-understood static fermion properties of QED. In QCD, particularly QCD sum-rule applications (Shifman *et al.*, 1979), such condensates characterize the vacuum expectation values of normal ordered products of fields (Pascual and Tarrach, 1984). For example (Elias *et al.*, 1988; Yndurain, 1989), the quark–antiquark condensate characterizes the QCD vacuum expectation value

$$\langle 0 | \psi_i^a(y) \bar{\psi}_j^b(z) | 0 \rangle = -\frac{\delta^{ab}}{3} \langle \bar{q}q \rangle \sum_{r=0}^{\infty} C_r [-im\gamma^\mu (y_\mu - z_\mu)]_{ij}^r \tag{1.1a}$$

$$C_r = \begin{cases} \frac{1}{(r/2)! [(r+2)/2]! 2^{r+2}}, & r \text{ even} \\ \frac{1}{[(r-1)/2]! [(r+3)/2]! 2^{r+2}}, & r \text{ odd} \end{cases} \tag{1.1b}$$

i, j are Dirac indices and a, b are color indices. In principle, every Feynman amplitude whose Wick–Dyson expansion of time-ordered fields contains a term in which a fermion and antifermion field are contracted to form a propagator,

$$\underbrace{\psi_i^a(y) \bar{\psi}_j^b(z)} = \langle 0 | T \psi_i^a(y) \bar{\psi}_j^b(z) | 0 \rangle = \int \frac{d^4 p}{(2\pi)^4} e^{-ip \cdot (y-z)} [S_F(p)]_{ij}^{ab} \tag{1.2}$$

also contains a not fully contracted term containing (1.1). It is precisely through such terms, methodologically, that QCD-vacuum condensates are introduced into the field-theoretic side of sum-rule calculations (Pascual and Tarrach, 1984). It should be noted that the *only* signature of nonabelian physics in the derivation of (1.1) is an overall color summation factor of 3 ($=\delta^{aa}$) in the denominator, a factor that can easily be absorbed in a redefinition of the fermion–antifermion condensate $\langle ff \rangle$ for the abelian case. [We will define $\langle ff \rangle$ henceforth such that $\delta^{ab} \langle \bar{q}q \rangle \rightarrow \langle ff \rangle$ in (1.1a).] The result (1.1) can be otherwise understood as a solution to the free Dirac equation with the condensate entering through an appropriately chosen initial condition (Yndurain, 1989). This nonzero condensate is a reflection of the nonperturbative content of the vacuum. Normal-ordered fields necessarily annihilate a purely-perturbative vacuum, which is why

vacuum expectation values like the left-hand side of (1.1) are *not* incorporated into standard QED calculations, but *are* incorporated into the field-theoretic content of QCD sum rules.

In this paper, we address whether the explicit fermion–antifermion condensate contribution to the anomalous QED magnetic moment of a fermion field is calculable through use of field-theoretic techniques for this contribution (Bagan *et al.*, 1993, 1994) adapted from QCD sum-rule applications. We are motivated to examine this particular property because it can be easily extracted from the leading corrections to the unrenormalized electromagnetic vertex function without reference to self-energy, vacuum polarization, or bremsstrahlung graphs that enter into the determination of electromagnetic form-factor slopes.

In Section 2 we provide a brief methodological review of how the anomalous magnetic moment $\mathcal{H}F_2(0)$ of a Dirac fermion with mass m is calculated in (purely) perturbative QED. In Section 3, we modify this calculation, as indicated above, by including the contribution of vacuum expectation values (1.1) in the Wick–Dyson expansion of the unrenormalized QED vertex amplitude. We find that the real part of the $\langle ff \rangle$ contribution to $\mathcal{H}F_2(0)$ vanishes, but that an imaginary part develops for q^2 between zero and $4m^2$ which diverges in the $q^2 \rightarrow 0$ limit.

These results are discussed in Section 4. They are first shown to be gauge parameter independent provided the same fermion mass characterizes (1.1) and (1.2), results suggestive of a dynamical rather than a Lagrangian origin for the common fermion mass. We then argue that a change in the kinematic threshold for the production of physical elementary particle states is the most sensible interpretation of the imaginary part obtained in Section 3, indicative of the production of Goldstone bosons anticipated from the dynamical breakdown of chiral symmetry ($\langle ff \rangle \neq 0$).

Up to this point in the paper, the question of condensate contributions to the anomalous magnetic moment has been posed entirely in the abstract. In Section 4, we discuss the applicability of the results of Section 3 to quarks, as fundamental Dirac fermions which form nonzero $\langle \bar{q}q \rangle$ condensates whose contribution to $\mathcal{H}F_2(0)$ is precisely of the type investigated in Section 3. Although quarks are confined, their QED magnetic moments and form factors are nevertheless of phenomenological interest for extracting baryon magnetic moments (Beg *et al.*, 1964; Perkins, 1987) and form-factor behavior. In the *absence* of condensate contributions, $\mathcal{H}F_2(q^2)$ develops an imaginary part when $q^2 > 4m^2$, corresponding to the production of on-shell fermion–antifermion pairs. The $\langle \bar{q}q \rangle$ contributions are seen to reduce this threshold to $q^2 = 0$, which can most easily be understood (assuming m is dynamical) to correspond to the kinematical production of massless pions, the Goldstone bosons of the chiral symmetry breaking

whose order-parameter manifestation is the $\langle \bar{q}q \rangle$ condensate itself. Thus, the change in the onset of an imaginary part in $\mathcal{H}F_2(q^2)$ may reflect the transition of QCD to low-energy hadronic physics.

2. THE ELECTROMAGNETIC VERTEX CORRECTION: A METHODOLOGICAL REVIEW

The purely perturbative three-point Green’s function (Fig. 1) containing the truncated fermion–antifermion-photon vertex Green’s function $-iQe\Gamma^\sigma(p_2, p_1)$ is expressed in terms of Heisenberg fields $\psi_i, \bar{\psi}_j, A_\mu$ as follows:

$$\begin{aligned}
 & [G_\mu(p_2, p_1)]_{il} \\
 &= \left[\frac{-i}{(p_2 - p_1)^2} \left(g_{\mu\sigma} - (1 - \xi) \frac{(p_2 - p_1)_\mu (p_2 - p_1)_\sigma}{(p_2 - p_1)^2} \right) \right] \left[\frac{i}{p_2 - m} \right]_{ij} \\
 &\times \left[\frac{i}{p_1 - m} \right]_{kl} [-ieQ\Gamma_{jk}^\sigma(p_2, p_1)] \\
 &= \int d^4x' \int d^4y' \langle 0|T\psi_i(x')A_\mu(0)\bar{\psi}_l(y')|0\rangle_{\text{Heis}} e^{+ip_2x'} e^{-ip_1y'} \quad (2.1)
 \end{aligned}$$

where $i - l$ are Dirac indices. The unrenormalized one-loop vertex correction Λ^μ within $\Gamma^\mu (= \gamma^\mu + \Lambda^\mu + \dots)$ is obtained via a Wick–Dyson expansion of the vacuum expectation value in (2.1) evaluated in the interaction picture (Dirac indices have been dropped):

$$\begin{aligned}
 & \langle 0|T\psi(x')A_\mu(0)\bar{\psi}(y')|0\rangle_{\text{Heis}} \\
 &= \langle 0|T\psi(x') \exp \left[-iQe \int d^4w \bar{\psi}(w)\gamma^\tau\psi(w)A_\tau(w) \right] A_\mu(0)\bar{\psi}(y')|0\rangle \quad (2.2)
 \end{aligned}$$

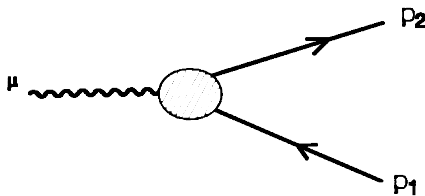


Fig. 1. The fermion–antifermion photon Green’s function.

The one-loop correction to G_μ in (2.1) is then found to be (Fig. 2)

$$\begin{aligned}
 & [\Delta G_\mu(p_2, p_1)]^{1\text{-loop}} \\
 &= (-iQe)^3 \int d^4x' \int d^4y' e^{+ip_2 \cdot x'} e^{-ip_1 \cdot y'} \\
 &\times \left[\int d^4x \int d^4y \int d^4z \langle 0|T\Psi(x')\bar{\Psi}(x)|0\rangle \gamma^\tau \right. \\
 &\times \langle 0|T\Psi(x)\bar{\Psi}(y)|0\rangle \gamma^\sigma \langle 0|T\Psi(y)\bar{\Psi}(z)|0\rangle \gamma^\rho \\
 &\times \langle 0|T\Psi(z)\bar{\Psi}(y')|0\rangle \langle 0|TA_\mu(0)A_\sigma(y)|0\rangle \\
 &\left. \times \langle 0|TA_\tau(x)A_\rho(z)|0\rangle \right] \tag{2.3}
 \end{aligned}$$

where the term in square brackets on the rhs is just the fully contracted third-order term in the Wick–Dyson expansion of (2.2). Equation (2.3) can be evaluated by explicit use of the configuration-space fermion and photon propagators

$$\langle 0|T\Psi(x)\bar{\Psi}(y)|0\rangle = i \int \frac{d^4q}{(2\pi)^4} e^{-iq \cdot (x-y)} \frac{1}{\not{q} - m} \tag{2.4}$$

$$\langle 0|TA_\tau(x)A_\rho(z)|0\rangle = -i \int \frac{d^4k}{(2\pi)^4} e^{-ik \cdot (x-z)} \frac{g_{\tau\rho}}{k^2} \tag{2.5}$$

We have omitted Dirac and color indices from (2.4), as well as the gauge-dependent longitudinal term from (2.5), as it does not contribute to the vertex function. Upon substitution of configuration-space propagators (2.4) and (2.5) into (2.3) and integration over the configuration-space variables $\{x', y', x, y,$

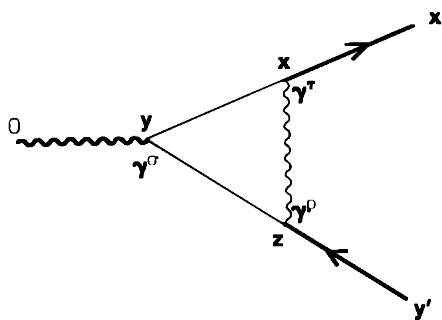


Fig. 2. The one-loop purely perturbative contribution to the fermion–antifermion-photon Green’s function in configuration space.

$z\}$, one obtains a string of delta functions which, when integrated over, yield the usual momentum-space Feynman propagator functions:

$$\begin{aligned}
 & [\Delta G_{\mu}(p_2, p_1)]_{il}^{-\text{loop}} \\
 &= (-iQe)^3 \int \frac{d^4 q_1}{(2\pi)^4} \int \frac{d^4 q_2}{(2\pi)^4} \int \frac{d^4 q_3}{(2\pi)^4} \int \frac{d^4 q_4}{(2\pi)^4} \int \frac{d^4 k_1}{(2\pi)^4} \int \frac{d^4 k_2}{(2\pi)^4} \\
 &\times \left(\frac{i}{q_1 - m} \right) \gamma^\tau \left(\frac{i}{q_2 - m} \right) \gamma^\sigma \left(\frac{i}{q_3 - m} \right) \gamma^\rho \left(\frac{i}{q_4 - m} \right) \\
 &\times \left(-\frac{ig_{\mu\sigma}}{k_1^2} \right) \left(-\frac{ig_{\nu\rho}}{k_2^2} \right) (2\pi)^{20} \delta^4(q_1 - p_2) \delta^4(q_4 - p_1) \\
 &\times \delta^4(p_2 - q_2 - k_2) \delta^4(q_2 - q_3 + k_1) \delta^4(q_3 - p_1 + k_2) \\
 &= \left[-\frac{ig_{\mu\sigma}}{(p_2 - p_1)^2} \right] \left[\frac{i}{p_2 - m} \right]_{ij} \left[\frac{i}{p_1 - m} \right]_{kl} (-iQe) \\
 &\times \left\{ -\frac{i(Qe)^2}{(2\pi)^4} \int \frac{d^4 k_2}{k_2^2} \gamma^\tau \frac{(p_2 - k_2 + m)}{(p_2 - k_2)^2 - m^2} \gamma^\sigma \frac{(p_1 - k_2 + m)}{(p_1 - k_2)^2 - m^2} \gamma^\tau \right\}_{jk} \quad (2.6)
 \end{aligned}$$

Factorization of the external legs is explicit in the final line of (2.6). The curly bracketed expression in the final line of (2.6) corresponds to the unrenormalized vertex correction $[\Lambda^\sigma(p_2, p_1)]$. Specifically, one can define the unrenormalized vertex correction to be $\bar{u}(p_2) \Lambda^\mu(p_2, p_1) u(p_1)$, where $(q^\mu \equiv p_2^\mu - p_1^\mu)$

$$\Lambda^\mu(p_2, p_1) \equiv e^2 Q^2 \left[R(q^2) \gamma^\mu + \frac{2S(q^2)}{m} (p_1^\mu + p_2^\mu) \right] \quad (2.7)$$

such that the unrenormalized vertex is $-ieQ\Gamma^\mu \equiv -ieQ(\gamma^\mu + \Lambda^\mu(p_2, p_1))$ with eQ the electromagnetic fermion charge. This unrenormalized vertex can be expressed as follows in terms of the renormalized vertex form factors $F_1(q^2)$, $\mathcal{K}F_2(q^2)$:

$$\begin{aligned}
 & \bar{u}(p_2) [\gamma^\mu + \Lambda^\mu(p_2, p_1)] u(p_1) \\
 &= \bar{u}(p_2) [(1 + e^2 Q^2 [R(q^2) + 4S(q^2)]) \gamma^\mu - 2e^2 Q^2 S(q^2) i\sigma^{\mu\nu} q_\nu / m] u(p_1) \\
 &\equiv Z \bar{u}(p_2) [F_1(q^2) \gamma^\mu + i\sigma^{\mu\nu} q_\nu \mathcal{K}F_2(q^2) / 2m] u(p_1) \quad (2.8)
 \end{aligned}$$

The rescaling in the final line of (2.8) is accomplished through the renormalization condition that $F_1(0) = 1$, in which case the (divergent) constant Z is given to order e^2 by

$$Z = 1 + e^2 Q^2 (R(0) + 4S(0)) \quad (2.9)$$

To leading order in e^2 , one then finds that

$$F_1(q^2) = 1 + e^2 Q^2 [(R'(0) + 4S'(0))q^2 + \mathcal{O}(q^4)] + \mathcal{O}(e^4) \quad (2.10)$$

$$\mathcal{K}F_2(q^2) = -4e^2 Q^2 S(q^2) + \mathcal{O}(e^4) \quad (2.11)$$

The $q^2 \rightarrow 0$ limit of equation (2.11) gives the $\mathcal{O}(\alpha)$ anomalous magnetic moment of QED, which (in contrast to F_1) devolves solely from the vertex correction and is insensitive to additional (vacuum-polarization, self-energy, and bremsstrahlung) diagrams.

The purely perturbative contribution to this quantity can be extracted by straightforward methods from the unrenormalized vertex correction in (2.6):

$$\begin{aligned} [\Lambda^\mu(p_2, p_1)]^{\text{pert}} &= -i(Qe)^2 \int \frac{d^4 k}{(2\pi)^4} \frac{\gamma_\tau (\not{p}_2 - k + m) \gamma^\mu (\not{p}_1 - k + m) \gamma^\tau}{k^2 [(p_2 - k)^2 - m^2] [(p_1 - k)^2 - m^2]} \\ &= -i(Qe)^2 \{ [-2\not{p}_1 \gamma^\mu \not{p}_2 + 4m(p_1^\mu + p_2^\mu) - 2m^2 \gamma^\mu] I(p_2, p_1) \\ &\quad + [2\gamma^\rho \gamma^\mu \not{p}_2 + 2\not{p}_1 \gamma^\mu \gamma^\rho - 8m g^{\mu\rho}] I_\rho(p_2, p_1) \\ &\quad - 2\gamma^\rho \gamma^\mu \gamma^\sigma I_{\rho\sigma}(p_2, p_1) \} \end{aligned} \quad (2.12)$$

The integrals I , I_ρ , and $I_{\rho\sigma}$ are respectively defined by

$$\begin{aligned} &[I(p_2, p_1); I_\rho(p_2, p_1); I_{\rho\sigma}(p_2, p_1)] \\ &\equiv \int \frac{d^n k}{(2\pi)^n} \frac{[1; k_\rho; k_\rho k_\sigma]}{(k^2 - \varepsilon^2) [(k - p_2)^2 - m^2] [(k - p_1)^2 - m^2]} \end{aligned} \quad (2.13)$$

In (2.13), ultraviolet divergences are regulated via dimensional regularization, and infrared divergences are regulated via the ‘‘photon mass’’ ε . These integrals are easily evaluated by standard methods. If we evaluate these integrals in terms of a set of constants $a_{0,1}$, $b_{0,1}$, . . . , such that

$$I(p_2, p_1) = i\{a_0/m^2 + a_1 q^2/m^4 + \mathcal{O}(q^4)\} \quad (2.14)$$

$$I_\rho(p_2, p_1) = i[(p_{1\rho} + p_{2\rho})/m^2] \{b_0 + b_1 q^2/m^2 + \mathcal{O}(q^4)\} \quad (2.15)$$

$$\begin{aligned} I_{\rho\sigma}(p_2, p_1) &= i g_{\rho\sigma} \{c_0 + c_1 q^2/m^2 + \mathcal{O}(q^4)\} \\ &\quad + i[(p_{1\rho} p_{1\sigma} + p_{2\rho} p_{2\sigma})/m^2] \{d_0 + d_1 q^2/m^2 + \mathcal{O}(q^4)\} \\ &\quad + i[(p_{1\rho} p_{2\sigma} + p_{2\rho} p_{1\sigma})/m^2] \{e_0 + e_1 q^2/m^2 + \mathcal{O}(q^4)\} \end{aligned} \quad (2.16)$$

we can then express the vertex correction (2.12) in the form (2.7):

$$\begin{aligned} R(q^2) &= [4a_0 - 16b_0 + 4c_0 + 4d_0 + 4e_0] \\ &\quad + (q^2/m^2)[-2a_0 + 4a_1 + 4b_0 - 16b_1 + 4c_1 \\ &\quad + 4d_1 - 2e_0 + 4e_1] + \mathcal{O}(q^4/m^4) \end{aligned} \quad (2.17)$$

$$S(q^2) = [2b_0 - 2d_0 - 2e_0] + (q^2/m^2) [2b_1 - 2d_1 - 2e_1] + \mathcal{O}(q^4/m^4) \quad (2.18)$$

The set of constants is determined via explicit evaluation of the integrals in (2.13):

$$a_0 = \frac{1}{32\pi^2} \ln\left(\frac{\xi^2}{m^2}\right), \quad a_1 = \frac{1}{192\pi^2} \left[1 + \ln\left(\frac{\xi^2}{m^2}\right) \right] \quad (2.19)$$

$$b_0 = -\frac{1}{32\pi^2}, \quad b_1 = -\frac{1}{192\pi^2} \quad (2.20)$$

$$c_0 = \frac{1}{64\pi^2} \left[-\frac{2}{n-4} - \gamma_E - \ln\left(\frac{m^2}{4\pi\mu^2}\right) + \frac{1}{2} \right], \quad c_1 = \frac{1}{384\pi^2} \quad (2.21)$$

$$d_0 = -\frac{1}{96\pi^2}, \quad d_1 = -\frac{1}{640\pi^2} \quad (2.22)$$

$$e_0 = -\frac{1}{192\pi^2}, \quad e_1 = -\frac{1}{960\pi^2} \quad (2.23)$$

As is evident from comparison of (2.11) and (2.18), the anomalous magnetic moment is clearly finite,

$$\mathcal{H}F_2(0) = -8e^2Q^2(b_0 - d_0 - e_0) = e^2Q^2/8\pi^2 \quad (2.24)$$

which is the famous result of Schwinger and Feynman (Schwinger, 1948; Feynman, 1949). The F_1 form-factor slope in the vertex correction (2.10) is also a classical result of perturbative QED,

$$\begin{aligned} F_1'(0) &= e^2Q^2[R'(0) + 4S'(0)] \\ &= \frac{e^2Q^2}{m^2} [-2a_0 + 4a_1 + 4b_0 - 8b_1 + 4c_1 - 4d_1 - 2e_0 - 4e_1] \\ &= -\frac{e^2Q^2}{24\pi^2m^2} \left[\frac{3}{4} + \ln\frac{\xi^2}{\mu^2} \right] \end{aligned} \quad (2.25)$$

although the removal of the photon mass from the physically measurable form-factor slope entails careful consideration of soft external-photon (bremsstrahlung) diagrams (Block and Nordsieck, 1937; Yennie *et al.*, 1955).

3. $\langle \bar{f}f \rangle$ CONTRIBUTION TO $\mathcal{K}F_2(q^2)$

In Fig. 3a [3b], we replace the perturbative configuration-space propagator $\langle 0|T\psi(x)\psi(y)|0\rangle$ [$\langle 0|T\psi(y)\psi(z)|0\rangle$] of equation (2.3) with the “nonperturbative propagator” (Yndurain, 1989) $\langle 0|\psi(x)\psi(y):|0\rangle$ [$\langle 0|\psi(y)\psi(z):|0\rangle$], the vacuum-expectation value of the normal-ordered pair of fermion fields that also arises from the Wick–Dyson expansion. Such vacuum-expectation values are routinely disregarded in purely perturbative calculations, in which the vacuum is fully annihilated by Fock-space annihilation operators, but must be taken into consideration if the vacuum has nonperturbative content (Pascual and Tarrach, 1984). The configuration-space nonperturbative propagator necessarily entails a replacement of (2.4) with the following expression (Bagan *et al.*, 1994; see also Elias *et al.*, 1988; Yndurain, 1989):

$$\langle 0|\psi(x)\bar{\psi}(y):|0\rangle = \int d^4k e^{-ik(x-y)}(\gamma \cdot k + m)\mathcal{F}(k) \tag{3.1a}$$

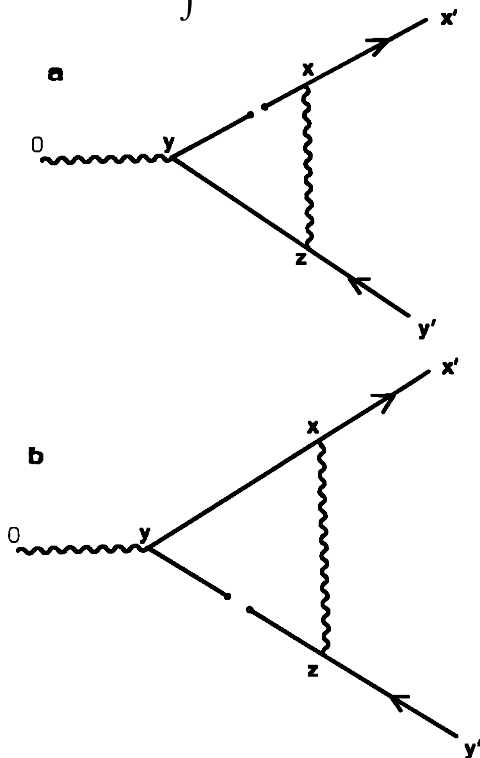


Fig. 3. The leading fermion–antifermion condensate contributions to the fermion–antifermion-photon Green’s function in configuration space, with nonperturbative propagators replacing internal fermion lines as indicated in (3.2) and (3.3).

where

$$\int d^4k \mathcal{F}(k)e^{-ik \cdot x} \equiv -\langle \bar{f}f \rangle J_1(m\sqrt{x^2})/(6m^2\sqrt{x^2}) \tag{3.1b}$$

with $\langle \bar{f}f \rangle$ identified as the (appropriately normalized) fermion–antifermion condensate of (1.1). As is evident from a comparison of (3.1a) to (2.4), the momentum-space expressions for the nonperturbative propagator contributions to Fig. 3a and Fig. 3b respectively entail the following alterations (Bagan *et al.*, 1994) in the final line of (2.6):

$$\text{Fig. 3a: } \frac{1}{(k_2 - p_1)^2 - m^2} \rightarrow -i(2\pi)^4 \mathcal{F}(k_2 - p_1) \tag{3.2}$$

$$\text{Fig. 3b: } \frac{1}{(k_2 - p_2)^2 - m^2} \rightarrow -i(2\pi)^4 \mathcal{F}(k_2 - p_2) \tag{3.3}$$

(The contribution vanishes from the graph in which both fermion internal lines are simultaneously altered.) The net effect of these changes is to reproduce the Feynman amplitude given in (2.12), but with the Feynman integrals (2.13) altered as follows:

$$\begin{aligned} & \int \frac{d^n k}{(2\pi)^n} \frac{[1; k_\rho; k_\rho k_\sigma]}{(k^2 - \varepsilon^2)[(k - p_2)^2 - m^2][(k - p_1)^2 - m^2]} \\ & \rightarrow -i \int \frac{d^4 k [1; k_\rho; k_\rho k_\sigma] \mathcal{F}(k - p_1)}{(k^2 - \varepsilon^2)[(k - p_2)^2 - m^2]} - i \int \frac{d^4 k [1; k_\rho; k_\rho k_\sigma] \mathcal{F}(k - p_2)}{(k^2 - \varepsilon^2)[(k - p_1)^2 - m^2]} \end{aligned} \tag{3.4}$$

We have returned to explicit use of four-dimensional integration because the new integrals are all UV-finite.

Thus we retain the form of the amplitude (2.12), but with the integrals $I, I_\rho, I_{\rho\sigma}$ now being given (after a trivial shift of integration variable) by

$$\begin{aligned} I(p_2, p_1) = & -i \int \frac{d^4 k \mathcal{F}(k)}{[(k + p_1)^2 - \varepsilon^2][(k + p_1 - p_2)^2 - m^2]} \\ & -i \int \frac{d^4 k \mathcal{F}(k)}{[(k + p_2)^2 - \varepsilon^2][(k + p_2 - p_1)^2 - m^2]} \end{aligned} \tag{3.5}$$

$$\begin{aligned}
 I_{\rho}(p_2, p_1) &= -i \int \frac{d^4k (k_{\rho} + p_{1\rho})\mathcal{F}(k)}{[(k + p_1)^2 - \varepsilon^2][(k + p_1 - p_2)^2 - m^2]} \\
 &\quad -i \int \frac{d^4k (k_{\rho} + p_{2\rho})\mathcal{F}(k)}{[(k + p_2)^2 - \varepsilon^2][(k + p_2 - p_1)^2 - m^2]} \quad (3.6)
 \end{aligned}$$

$$\begin{aligned}
 I_{\rho\sigma}(p_2, p_1) &= -i \int \frac{d^4k (k_{\rho} + p_{1\rho})(k_{\sigma} + p_{1\sigma})\mathcal{F}(k)}{[(k + p_1)^2 - \varepsilon^2][(k + p_1 - p_2)^2 - m^2]} \\
 &\quad -i \int \frac{d^4k (k_{\rho} + p_{2\rho})(k_{\sigma} + p_{2\sigma})\mathcal{F}(k)}{[(k + p_2)^2 - \varepsilon^2][(k + p_2 - p_1)^2 - m^2]} \quad (3.7)
 \end{aligned}$$

Using parametrizations analogous to (2.14)–(2.16),

$$I(p_2, p_1) = i\mathcal{A}(q^2)/m^2 \quad (3.8)$$

$$I_{\rho}(p_2, p_1) = i[(p_{1\rho} + p_{2\rho})/m^2]\mathcal{B}(q^2) \quad (3.9)$$

$$\begin{aligned}
 I_{\rho\sigma}(p_2, p_1) &= ig_{\rho\sigma}\mathcal{C}(q^2) \\
 &\quad + i[(p_{1\rho}p_{1\sigma} + p_{2\rho}p_{2\sigma})/m^2]\mathcal{D}(q^2) \\
 &\quad + i[(p_{1\rho}p_{2\sigma} + p_{2\rho}p_{1\sigma})/m^2]\mathcal{E}(q^2) \quad (3.10)
 \end{aligned}$$

we proceed analogously to the derivation of (2.18) and find that the $\langle \bar{f}f \rangle$ contribution to $S(q^2)$ in (2.7) is now given to one-loop order by

$$\Delta S(q^2) = 2\mathcal{B}(q^2) - 2\mathcal{D}(q^2) - 2\mathcal{E}(q^2) \quad (3.11)$$

As in (2.24), the fermion–antifermion condensate contribution to the anomalous magnetic moment is then found to be

$$\Delta\mathcal{H}F_2(0) = -8(e^2Q^2)(\mathcal{B}(0) - \mathcal{D}(0) - \mathcal{E}(0)) \quad (3.12)$$

To proceed further, we need to evaluate the integrals (3.6) and (3.7) that determine the explicit functions $\mathcal{B}(q^2)$, $\mathcal{D}(q^2)$, and $\mathcal{E}(q^2)$ of (3.9) and (3.10). To evaluate $\mathcal{B}(q^2)$, we need to evaluate the integrals $I_{\rho}(p_2, p_1)$ in equation (3.6). Using a Feynman-parameter combination of the propagator denominators, we find for on-shell momenta ($p_1^2 = p_2^2 = m^2$) that

$$\begin{aligned}
 I_{\rho}(p_2, p_1) &= -i \int_0^1 dz \int \frac{d^4k (k_{\rho} + p_{1\rho})\mathcal{F}(k)}{\{[k - (p_2z - p_1)]^2 - m^2z^2 - \varepsilon^2(1 - z)\}^2} \\
 &\quad -i \int_0^1 dz \int \frac{d^4k (k_{\rho} + p_{2\rho})\mathcal{F}(k)}{\{[k - (p_{1z} - p_2)]^2 - m^2z^2 - \varepsilon^2(1 - z)\}^2} \quad (3.13)
 \end{aligned}$$

We will drop the photon mass ε^2 , as any infrared divergences that might arise in the anomalous magnetic moment [as opposed to $F_1(q^2)$] cannot be removed by bremsstrahlung corrections. The integrals in (3.13) can be expressed in terms of the integrals (A.6) and (A.7) of the Appendix, with $p \equiv p_{1z} - p_2$ or $p_{2z} - p_1$, and with $\mu = mz$. For on-shell momenta, both definitions of p lead to $p^2 = m^2(1 - z)^2 + q^2z$ with $q^2 = (p_2 - p_1)^2$. The results we obtain are valid only for $q^2 > 0$; the requirement that $p^2 > 0$, as discussed immediately following (A.9), necessarily implies $q^2 > 0$, as z in $p^2 = m^2(1 - z)^2 + q^2z$ ranges over values between zero and one. We then see from (3.13) that

$$\begin{aligned}
 I_p(p_2, p_1) = & -ip_{1\rho} \int_0^1 dz R_3(p_{2z} - p_1, mz) \\
 & -ip_{2\rho} \int_0^1 dz R_3(p_{1z} - p_2, mz) \\
 & -i \int_0^1 dz \int \frac{d^4k k_\rho \mathcal{F}(k)}{\{[k - (p_{2z} - p_1)]^2 - m^2z^2\}^2} \\
 & -i \int_0^1 dz \int \frac{d^4k k_\rho \mathcal{F}(k)}{\{[k - (p_{1z} - p_2)]^2 - m^2z^2\}^2} \quad (3.14)
 \end{aligned}$$

where, from the Appendix, we define

$$R_3(p, \mu) \equiv \int \frac{d^4k \mathcal{F}(k)}{[(k - p)^2 - \mu^2]^2} \quad (3.15a)$$

$$R_2(p, \mu) \equiv \int \frac{d^4k \mathcal{F}(k)}{[(k - p)^2 - \mu^2]} \quad (3.15b)$$

The remaining integrals in (3.14) are of the form

$$\int \frac{d^4k k^\rho \mathcal{F}(k)}{[(k - p)^2 - \mu^2]^2} = A(p^2)p^\rho \quad (3.16)$$

If we contract p_ρ into both sides of (3.16) and use the identities $p \cdot k = -\frac{1}{2}[(k - p)^2 - \mu^2] + k^2/2 + p^2/2 - \mu^2/2$ and $k^2\mathcal{F}(k) = m^2\mathcal{F}(k)$ [equation (A.4)], we find that

$$A(p^2) = -\frac{1}{2p^2} R_2(p, \mu) + \frac{m^2 + p^2 - \mu^2}{2p^2} R_3(p, \mu) \quad (3.17)$$

Noting that $\mu^2 = m^2z^2, p^2 = m^2(1 - z)^2 + q^2z$, we then find that

$$\begin{aligned}
 I_p(p_2, p_1) &= -ip_{1\rho} \int_0^1 dz R_3(p_2z - p_1, mz) \\
 &\quad - ip_{2\rho} \int_0^1 dz R_3(p_1z - p_2, mz) \\
 &\quad - i \int_0^1 dz (p_{2\rho z} - p_{1\rho}) \left[-\frac{1}{2[m^2(1 - z)^2 + q^2z]} R_2(p_2z - p_1, mz) \right. \\
 &\quad \left. + \frac{2m^2(1 - z) + q^2z}{2[m^2(1 - z)^2 + q^2z]} R_3(p_2z - p_1, mz) \right] \\
 &\quad - i \int_0^1 dz (p_{1\rho z} - p_{2\rho}) \left[-\frac{1}{2[m^2(1 - z)^2 + q^2z]} R_2(p_1z - p_2, mz) \right. \\
 &\quad \left. + \frac{2m^2(1 - z) + q^2z}{2[m^2(1 - z)^2 + q^2z]} R_3(p_1z - p_2, mz) \right] \tag{3.18}
 \end{aligned}$$

We see from (A.6) and (A.7) of the Appendix that $R_2(p, \mu)$ and $R_3(p, \mu)$ depend on p only through p^2 :

$$R_2(p_1z - p_2, mz) = R_2(p_2z - p_1, mz) \equiv R_2[z] \tag{3.19a}$$

$$R_3(p_1z - p_2, mz) = R_3(p_2z - p_1, mz) \equiv R_3[z] \tag{3.19b}$$

and from (3.17) we find that

$$\begin{aligned}
 A[z] &\equiv A(m^2(1 - z)^2 + q^2z) \\
 &= \frac{[2m^2(1 - z) + q^2z]R_3[z] - R_2[z]}{2[m^2(1 - z)^2 + q^2z]} \tag{3.20}
 \end{aligned}$$

By comparing (3.18) to (3.9), we obtain

$$\mathfrak{B}(q^2) = m^2 \int_0^1 dz [(1 - z)A[z] - R_3[z]] \tag{3.21}$$

To find $\mathfrak{D}(q^2)$ and $\mathfrak{E}(q^2)$ in (3.10), we combine the propagator denominators of (3.7) to obtain

$$\begin{aligned}
I_{\rho\sigma} &= -i \int_0^1 dz \int d^4k \frac{(k_\rho + p_{1\rho})(k_\sigma + p_{1\sigma})\mathcal{F}(k)}{\{[k - (p_2z - p_1)]^2 - m^2z^2\}^2} \\
&\quad -i \int_0^1 dz \int d^4k \frac{(k_\rho + p_{2\rho})(k_\sigma + p_{2\sigma})\mathcal{F}(k)}{\{[k - (p_1z - p_2)]^2 - m^2z^2\}^2} \\
&= +i(p_{1\rho}p_{1\sigma} + p_{2\rho}p_{2\sigma}) \int_0^1 dz [2A[z] - R_3[z]] \\
&\quad -i(p_{1\rho}p_{2\sigma} + p_{2\rho}p_{1\sigma}) \int_0^1 dz [2zA[z]] \\
&\quad -i \int_0^1 \frac{dz k_\rho k_\sigma \mathcal{F}(k)}{\{[k - (p_2z - p_1)]^2 - m^2z^2\}^2} \\
&\quad -i \int_0^1 \frac{dz k_\rho k_\sigma \mathcal{F}(k)}{\{[k - (p_1z - p_2)]^2 - m^2z^2\}^2} \tag{3.22}
\end{aligned}$$

The remaining integrals in (3.22) are of the form

$$\int \frac{d^4k k_\rho k_\sigma \mathcal{F}(k)}{\{(k - p)^2 - \mu^2\}^2} = X(p^2)g_{\rho\sigma} + Y(p^2)p_\rho p_\sigma \tag{3.23}$$

in which case we see from comparison of (3.22) to (3.10) that

$$\mathcal{D}(q^2) = m^2 \int_0^1 dz [2A[z] - R_3[z] - (1 + z^2)Y[z]] \tag{3.24}$$

$$\mathcal{E}(q^2) = +m^2 \int_0^1 dz [-2zA[z] + 2zY[z]] \tag{3.25}$$

where

$$Y[z] \equiv Y(m^2(1 - z)^2 + q^2z) \tag{3.26}$$

Thus we see through comparison of (3.11) to (3.21), (3.24), and (3.25) that

$$\Delta S(q^2) = 2m^2 \int_0^1 dz [(1 - z)^2Y[z] - (1 - z)A[z]] \tag{3.27}$$

To determine $Y[z]$, we first note from (A.4) in the Appendix that contraction of $g^{\rho\sigma}$ into (3.23) yields the relation

$$m^2R_3(p, \mu) = 4X(p^2) + p^2Y(p^2) \tag{3.28}$$

Recalling that $p \cdot k = -\frac{1}{2}[(k - p)^2 - \mu^2] + \frac{1}{2}(m^2 + p^2 - \mu^2)$, we see that contraction of $p^\rho p^\sigma$ into (3.23) yields the relation

$$\begin{aligned} \frac{1}{4} R_1 - \frac{1}{2} (m^2 + p^2 - \mu^2) R_2(p, \mu) + \frac{1}{4} (m^2 + p^2 - \mu^2)^2 R_3(p, \mu) \\ = p^2 X(p^2) + (p^2)^2 Y(p^2) \end{aligned} \tag{3.29}$$

where $R_1 = \int d^4k \mathcal{F}(k) = -\langle \bar{f}f \rangle / 12m$, as shown in the Appendix. Given $\mu = mz$, $p^2 = m^2(1 - z)^2 + q^2z$, one can then solve (3.28) and (3.29) for $Y(p^2)$ to obtain the following:

$$\begin{aligned} Y[z] = & -\frac{1}{3[m^2(1 - z)^2 + q^2z]} \{m^2 R_3[z] \\ & - \frac{1}{(m^2(1 - z)^2 + q^2z)} [R_1 - 2(2m^2(1 - z) + q^2z) R_2[z] \\ & + [2m^2(1 - z) + q^2z]^2 R_3[z]]\} \end{aligned} \tag{3.30}$$

The integral (3.27) can be evaluated using the expressions (3.20) and (3.30) to express $A[z]$ and $Y[z]$ in terms of R_1 , $R_2[z]$, and $R_3[z]$. We note from (A.6) and (A.7) of the Appendix that for $p^2 = m^2(1 - z)^2 + q^2z$, $\mu^2 = m^2z^2$, and $4m^2 > q^2 > 0$,

$$R_2[z] = \frac{\langle \bar{f}f \rangle}{24m^3} \frac{[-2m^2(1 - z) - q^2z + iz\sqrt{4m^2q^2 - q^4}]}{m^2(1 - z)^2 + q^2z} \tag{3.31}$$

$$R_3[z] = \frac{\langle \bar{f}f \rangle}{24m^3[m^2(1 - z)^2 + q^2z]} \left[1 + i \frac{[2m^2(1 - z) + q^2z]}{z\sqrt{4m^2q^2 - q^4}} \right] \tag{3.32}$$

We note that R_2 and R_3 have developed imaginary parts when q^2 is between zero and $4m^2$; R_2 and R_3 are both real if $q^2 > 4m^2$. Such a branch cut between $q^2 = 0$ and $q^2 = 4m^2$ is also seen to occur in the quark–antiquark condensate contributions to two-point current-correlation functions (Bagan *et al.*, 1986; Elias *et al.*, 1994), and is discussed in detail in the section that follows. We find from substitution of (3.20) and (3.30)–(3.32) into (3.27) that

$$\begin{aligned} \Delta S(q^2) = 2m^2 \int_0^1 dz (1 - z) \left[-\frac{\langle \bar{f}f \rangle (1 - z)}{36m[m^2(1 - z)^2 + q^2z]^2} \right. \\ - \frac{[5m^2(1 - z)^2 + q^2z(1 - 4z)]}{6[m^2(1 - z)^2 + q^2z]^2} R_2[z] \\ \left. - \frac{[m^2q^2z(1 - z)(3 + 5z) + q^4z^2(1 + 2z)]}{6[m^2(1 - z)^2 + q^2z]^2} R_3[z] \right] \end{aligned} \tag{3.33}$$

Explicit evaluation of the real and imaginary parts of (3.33) for $0 < q^2 < 4m^2$ yields the following results:

$$\text{Re}[\Delta S(q^2)] = 0 \tag{3.34}$$

$$\text{Im}[\Delta S(q^2)] = -\frac{\langle \bar{f}f \rangle}{12m\sqrt{4m^2q^2 - q^4}} \tag{3.35}$$

where $\Delta \mathcal{H}F_2(q^2) = -4e^2 Q^2 \Delta S(q^2)$.

4. DISCUSSION

4.1. Gauge Invariance

In an arbitrary covariant gauge, the contribution of Fig. 3a is proportional to

$$\begin{aligned} & \int d^4k \mathcal{D}^{\tau\sigma}(k, \xi) \frac{\mathcal{F}(k - p_1)\gamma_\tau(k - p_2 - m)\gamma_\mu(k - p_1 - m)\gamma_\sigma}{[(k - p_2)^2 - m^2]} \\ & \equiv \Lambda^{(a)}(p_2, p_1) - (1 - \xi)\Lambda_\xi^{(a)}(p_2, p_1) \end{aligned} \tag{4.1}$$

as is evident from (3.3) and (2.6), where ξ is the photon-propagator gauge parameter

$$\mathcal{D}^{\tau\sigma}(k) = \frac{g^{\tau\sigma}}{k^2} - (1 - \xi) \frac{k^\tau k^\sigma}{k^4} \tag{4.2}$$

In (4.1), $\Lambda^{(a)}$ is the (Feynman-gauge) contribution we have already considered, and $\Lambda_\xi^{(a)}$ is the contribution arising from the second term in (4.2). Gauge-parameter independence is explicit provided $\bar{u}(p_2)\Lambda_\xi^{(a)}(p_2, p_1)u(p_1) = 0$, *i.e.*, provided the $k^\tau k^\sigma/k^4$ term in $\mathcal{D}^{\tau\sigma}$ does not contribute to the on-shell vertex correction. To demonstrate this, we consider

$$\begin{aligned} & \Lambda_\xi^{(a)}(p_2, p_1)u(p_1) \\ & = \int \frac{d^4k \mathcal{F}(k - p_1)k(k - p_2 - m)\gamma_\mu(k - p_1 - m)k}{k^4[(k - p_2)^2 - m^2]} u(p_1) \\ & \xrightarrow{k-p_1 \rightarrow k} \int \frac{d^4k (k + p_1)(k + p_1 - p_2 - m)\gamma_\mu \mathcal{F}(k)(k - m)(k + p_1)u(p_1)}{[(k + p_1)^2]^2[(k + p_1 - p_2)^2 - m^2]} \end{aligned} \tag{4.3}$$

$\mathcal{F}(k)$ is a Dirac scalar that can be moved past gamma-matrices—we have moved it to the right in (4.3) in order to focus on the factors immediately preceding $u(p_1)$:

$$\begin{aligned}
 & \mathcal{F}(k)(k - m)(k + p_1)u(p_1) \\
 &= \mathcal{F}(k)(k^2 - m^2)u(p_1) \\
 &= 0
 \end{aligned}
 \tag{4.4}$$

The second to last line of (4.4) is a consequence of $p_1 u(p_1) = m u(p_1)$, and the final line is a consequence of $k^2 \mathcal{F}(k) = m^2 \mathcal{F}(k)$, as discussed in the Appendix [equation (A.4)]. Thus we see that $\Lambda_\xi^g(p_2, p_1)u(p_1) = 0$. A virtually identical argument shows the gauge-dependent contribution from Fig. 3b annihilates $\bar{u}(p_2)$ on-shell. Consequently, we see that the vertex correction $\bar{u}(p_2)\Lambda^\mu(p_2, p_1)u(p_1)$ is manifestly gauge-parameter independent: $\bar{u}(p_2)\Lambda_\xi^{(a)}(p_2, p_1)u(p_1) = 0$.

This demonstration of gauge-parameter independence, however, is contingent upon having *the same* fermion mass enter the perturbative and nonperturbative fermion propagators (2.4) and (3.1), as has been assumed throughout the previous section's calculation. Any attempt to distinguish between these masses will destroy the gauge-parameter independence of the result (*e.g.*, He, 1996). The gauge-parameter independence of electroweak two-point functions has similarly been shown (Ahmady *et al.*, 1989) to be contingent, for a given flavor, upon having the same fermion mass enter from nonperturbative vacuum expectation values (1.1) as appears in the corresponding fermion propagator function (1.2).

An entirely analogous situation arises in QCD when one considers the fermion–antifermion condensate contribution to the fermion two-point function. The apparent gauge-parameter dependence first seen for this contribution (Pascual and de Rafael, 1982) has been shown to disappear on-shell (Elias and Scadron, 1984) provided the mass that appears in the fermion propagator (2.4) is consistent with that appearing in the vacuum expectation value (1.1). Since this latter mass is necessarily dynamical, gauge-parameter independence suggests that the fermion mass appearing throughout the calculation of the previous section be understood to be dynamical rather than Lagrangian in origin (Elias and Scadron, 1984; Reinders and Stam, 1986), a reflection of the chiral noninvariance of the vacuum necessary for (1.1) to be nonzero (*i.e.*, for $\langle ff \rangle \neq 0$).

4.2. Interpretation of $\text{Im}(\Delta S(q^2))$

As noted in the previous section, the integrals $R_2[z]$ and $R_3[z]$ are seen to contribute an imaginary part to the vertex function when q^2 is between 0 and $4m^2$, behaviour that is also evident in the fermion–antifermion condensate contributions to two-point functions (Bagan *et al.*, 1986; Elias *et al.*, 1993). Although imaginary parts of Feynman amplitudes are signals of physical intermediate states, the region $0 < q^2 < 4m^2$ is clearly beneath the $q^2 =$

$4m^2$ kinematic threshold for the production of a physical fermion–antifermion (ff) pair. Nevertheless, this $0 < q^2 < 4m^2$ branch cut, when augmented by the purely perturbative ff -production branch cut beginning at $q^2 = 4m^2$, may be associated with the $q^2 = 0$ production threshold for Goldstone bosons associated with chiral symmetry breaking. The internal consistency of such a picture necessitates identification of m with a dynamical mass, as opposed to a mass that appears in the Lagrangian and that explicitly breaks Lagrangian chiral symmetry; *e.g.*, the pion is massless only in the limit of Lagrangian chiral symmetry (zero current-quark mass). As noted above, such a dynamical fermion mass is expected to arise from the chiral noninvariance of the QCD vacuum itself, and can be related directly to the $\langle ff \rangle$ order parameter characterizing chiral noninvariance (Politzer, 1976; Elias and Scadron, 1984). We have also seen above that such an interpretation is strongly suggested by gauge invariance. Thus it would appear that the nonzero imaginary part occurring in (3.35) may be a kinematic manifestation of the Goldstone theorem, suggesting that the theory now contains the zero-mass meson anticipated from a dynamical breakdown ($\langle ff \rangle \neq 0$) of Lagrangian chiral symmetry.

4.3. Quarks?

Although the field-theoretic calculations presented in this paper have been posed almost entirely in the abstract, there are clear reasons to explore their applicability to the electromagnetic properties of quarks. Quarks couple to both QED and QCD interactions. Even though the latter are deemed entirely responsible for the existence of $\langle \bar{q}q \rangle$ condensates, such condensates necessarily contribute to Feynman amplitudes from which quark electromagnetic properties are extracted.

As remarked in the Introduction, such properties are of evident interest to quark-model estimates of hadron properties. For example, the construction of proton and neutron magnetic moments from the magnetic moments of Dirac-fermion quark constituents (Beg *et al.*, 1964), one of the very earliest successes of the nonrelativistic quark model, is sensible only if the quark masses employed are vastly larger [$\mathcal{O}(300 \text{ MeV})$] than those masses anticipated from the QCD Lagrangian [$\mathcal{O}(5\text{--}10 \text{ MeV})$]. Thus, there is a clear phenomenological role for a dynamical mass in quark-model physics, even though such a larger mass may really represent the inverse length of a confinement radius.

Any application of the calculation presented in Section 3 to quark electromagnetic properties needs to recognize that QED and QCD cannot be treated in isolation. Not only is the chiral-noninvariant QCD vacuum the “standard-model” vacuum that quarks actually experience, suggesting the need to include $\langle \bar{q}q \rangle$ contributions; the photon exchanges of QED in isolation must also be

augmented by gluon exchanges. Consequently, the spin-1 internal lines of Fig 3 represent photons *and* gluons, entailing substitution of $[e^2Q^2 + g_s^2T(R)]$ for all factors of e^2Q^2 appearing in Section 3. This change becomes problematic in the $q^2 \rightarrow 0$ limit, a region believed to be inaccessible to perturbative QCD, though there exists evidence (and considerable prejudice) for a freezing out of the effective QCD coupling g_s to not overly large values at small momentum transfers (Mattingly and Stevenson, 1992; Stevenson, 1994; Ellis *et al.*, 1997; Baboukhadia *et al.*, 1997; Gardi and Karliner, 1998). Similarly, the nonabelian character of QCD leads to a vastly richer set of gluon bremsstrahlung graphs contributing to $F_1(q^2)$; it is precisely for this reason we have focused on $\mathcal{H}F_2$, a quantity insensitive to such graphs.

Subject to all these concerns, it is of interest to speculate on the applicability of the previous section's results to phenomenological quark properties. Naively, the result (3.34) would imply the absence of a $\langle\bar{q}q\rangle$ contribution to the quark magnetic moment, particularly if the divergence in the imaginary part at $q^2 = 0$, as seen from equation (3.35), is attributable to the production of zero-mass pions. *Moreover, an alteration in the kinematic production threshold from $4m^2$ to zero may reflect QCD's transition from a quark-gluon gauge theory to true low-energy hadronic physics.* As discussed above, m is understood in such a picture to be the $\mathcal{O}(300 \text{ MeV})$ dynamical quark mass characterizing the quark magneton in the static quark model. The success of such a picture appears not to be compromised by $\langle\bar{q}q\rangle$ effects, if (3.34) is taken at face value.

When $q^2 > 4m^2$, however, the quark-condensate contribution to $\mathcal{H}F_2(q^2)$ is entirely real:

$$\Delta S(q^2) = \frac{\langle\bar{q}q\rangle}{12m\sqrt{q^4 - 4m^2q^2}} \tag{4.5}$$

Even though m is understood here to be dynamical [$\langle\bar{q}q\rangle/m \sim m^2$ (Politzer, 1976; Elias and Scadron, 1984)], we have no explanation for the divergence in (4.5) as $q^2 \rightarrow 4m^2$ from above. One does see from (4.5) that the quark-condensate contribution to $\mathcal{H}F_2(q^2)$ goes like $1/q^2$ in the large- q^2 limit. Such behavior also characterizes (up to logarithms) the purely-perturbative contributions to $\mathcal{H}F_2(q^2)$ discussed in Section 2, and can be linked via quark-counting rules to the $1/Q^6$ behavior of the nucleon form factor $F_2^N(Q^2)$ (Brodsky and Lepage, 1989).

Finally, we point out that the *large- Q^2* behavior of the quark-condensate contribution (4.5) is not suppressed relative to the purely perturbative contribution. On dimensional grounds, one might expect an $\mathcal{O}[m\langle\bar{q}q\rangle/Q^4]$ quark-condensate contribution that is suppressed relative to the purely perturbative contribution at short distances. Such behavior is clearly unsupported by (4.5), which suggests that the nonperturbative order parameter $\langle\bar{q}q\rangle$ may leave footprints even in the deep-inelastic domain.

APPENDIX: FEYNMAN INTEGRALS OVER THE NONPERTURBATIVE PROPAGATOR FUNCTION $\mathcal{F}(k)$

The vacuum expectation value of a normal-ordered pair of condensing fermion–antifermion fields may be expressed as follows (Bagan *et al.*, 1993, 1994; Yndurain, 1989):

$$\begin{aligned} &\langle 0|:f(x)\bar{f}(0):|0\rangle \\ &= -\frac{\langle \bar{f}f \rangle}{6m^2} (i\gamma^\mu \partial_\mu + m)[J_1(m\sqrt{x^2})/\sqrt{x^2}] \\ &\equiv \int d^4k e^{-ikx} (\gamma^\mu k_\mu + m)\mathcal{F}(k) \end{aligned} \tag{A.1}$$

where \mathcal{F} is the Fourier transform

$$\int d^4k e^{-ik \cdot x} \mathcal{F}(k) = -\frac{\langle \bar{f}f \rangle}{6m^2} \frac{J_1(m\sqrt{x^2})}{\sqrt{x^2}} \tag{A.2}$$

which is well defined for causal Minkowskian separations ($x^2 > 0$). The normalization chosen for $\langle \bar{f}f \rangle$ is discussed in Section 1. The second line of (A.1) is a solution to the free-particle Dirac equation.

$$(i\gamma^\mu \partial_\mu - m)\langle 0|:f(x)\bar{f}(0):|0\rangle = 0 \tag{A.3}$$

Application of (A.3) to the final expression on the right-hand side of (A.1) implies that (Bagan *et al.*, 1994)

$$k^2\mathcal{F}(k) = m^2\mathcal{F}(k) \tag{A.4}$$

Let us now consider the following integrals arising in the text:

$$R_1 \equiv \int d^4k \mathcal{F}(k) = -\langle \bar{f}f \rangle / 12m \tag{A.5}$$

$$\begin{aligned} R_2(p, \mu) &\equiv \int \frac{d^4k \mathcal{F}(k)}{(p-k)^2 - \mu^2 + i|\epsilon|} = -\frac{\langle \bar{f}f \rangle}{24m^3 p^2} \\ &\times [p^2 + m^2 - \mu^2 - \sqrt{[p^2 - (m-\mu)^2][p^2 - (m+\mu)^2]}] \end{aligned} \tag{A.6}$$

$$\begin{aligned} R_3(p, \mu) &\equiv \int \frac{d^4k \mathcal{F}(k)}{[(p-k)^2 - \mu^2 + i|\epsilon|]^2} \\ &= \frac{\langle \bar{f}f \rangle}{24m^3 p^2} \left[1 - \frac{p^2 + m^2 - \mu^2}{[p^2 - (m-\mu)^2][p^2 - (m+\mu)^2]} \right] \end{aligned} \tag{A.7}$$

Equations (A.6) and (A.7) are demonstrably valid only for $p^2 > 0$, as discussed below.

The integral R_1 is obtained from the $x \rightarrow 0$ limit of (A.2). To evaluate the integral $R_2(p, \mu)$, we first utilize (A.4) to replace k^2 with m^2 , and we then utilize the positivity of $|\epsilon|$ to exponentiate the propagator:

$$\begin{aligned}
 R_2(p, \mu) &= -i \int_0^\infty d\eta e^{i\eta(p^2+m^2-\mu^2+i\epsilon|\epsilon|)} \int d^4k e^{-ik \cdot (2p\eta)} \mathcal{F}(k) \\
 &= \frac{i\langle \bar{f}f \rangle}{6m^2} \int_0^\infty d\eta e^{-\eta[|\epsilon| - i(p^2+m^2-\mu^2)]} J_1(2\eta m\sqrt{p^2}) / (2\eta\sqrt{p^2}) \quad (A.8)
 \end{aligned}$$

where the final line of (A.8) is obtained directly from (A.2) with x replaced by $2\eta p$. The resulting integral over η is evaluated through use of the tabulated integral (Gradshteyn and Ryzhik, 1980)

$$\int_0^\infty e^{-\eta\alpha} J_1(\eta\beta) d\eta/\eta = (\sqrt{\alpha^2 + \beta^2} - \alpha)/\beta, \quad (\text{Re } \alpha > |\text{Im } \beta|) \quad (A.9)$$

Since $\text{Re } \alpha$ is identified with the positive infinitesimal $|\epsilon|$ and β is identified with $2m\sqrt{p^2}$ in (A.8), p^2 must be positive. The following results are obtained for physical (Minkowskian) momenta:

$$\begin{aligned}
 R_2(p, \mu) &= \frac{i\langle \bar{f}f \rangle}{12m^2\sqrt{p^2}} \\
 &\times \left[-i \frac{\sqrt{(p^2 + m^2 - \mu^2)^2 - 4m^2p^2} - (p^2 + m^2 - \mu^2)}{2m\sqrt{p^2}} \right] \quad (A.10)
 \end{aligned}$$

which is easily rearranged to yield (A.6).

To evaluate the integral $R_3(p, \mu)$, we again utilize (A.4) to replace k^2 with m^2 and then exponentiate the propagator:

$$\begin{aligned}
 R_3(p, \mu) &= - \int_0^\infty d\eta \eta e^{i\eta(p^2+m^2-\mu^2+i\epsilon|\epsilon|)} \int d^4k \mathcal{F}(k) e^{-ik \cdot (2p\eta)} \\
 &= \frac{\langle \bar{f}f \rangle}{12m^2\sqrt{p^2}} \int_0^\infty d\eta \epsilon^{-\eta[|\epsilon| - i(p^2+m^2-\mu^2)]} J_1(2\eta m\sqrt{p^2}) \quad (A.11)
 \end{aligned}$$

The final integral in (A.11) is evaluated through use of the integral

$$\int_0^\infty e^{-\eta\alpha} J_1(\eta\beta) d\eta = \frac{\sqrt{\alpha^2 + \beta^2} - \alpha}{\beta\sqrt{\alpha^2 + \beta^2}} \quad (A.12)$$

by taking the partial derivative of both sides of (A.9) with respect to α . Using (A.12), which is again demonstrably valid ($|\epsilon| > |\text{Im } 2\eta\sqrt{p^2}|$) only for $p^2 > 0$, we easily obtain the result (A.7).

It is evident from these procedures that any Feynman integrals of the form

$$R_{N+1}(p, \mu) \equiv \int \frac{d^4k \mathcal{F}(k)}{[(p-k)^2 - \mu^2 + i|\varepsilon|]^N} \quad (\text{A.13})$$

can be evaluated through use of (A.2) and (A.4), exponentiation of the denominator, and successive differentiations of (A.9):

$$\begin{aligned} R_{N+1}(p, \mu) &= \frac{(-i)^N}{(N-1)!} \int_0^\infty d\eta \eta^{N-1} e^{-\eta[|\varepsilon| - i(p^2 + m^2 - \mu^2)]} \\ &\quad \times \int d^4k \mathcal{F}(k) e^{-ik \cdot (2p\eta)} \\ &= -\frac{(-i)^N}{(N-1)!} \frac{\langle \bar{f} f \rangle}{12m^2 \sqrt{p^2}} \int_0^\infty d\eta \eta^{N-2} e^{-\eta[|\varepsilon| - i(p^2 + m^2 - \mu^2)]} J_1(2m\eta \sqrt{p^2}) \\ &= \frac{i^N \langle \bar{f} f \rangle}{(N-1)! 12m^2 \sqrt{p^2}} \frac{\partial^{N-1}}{\partial \alpha^{N-1}} \left(\frac{\sqrt{\alpha^2 + \beta^2} - \alpha}{\beta} \right) \Bigg|_{\alpha = -i(p^2 + m^2 - \mu^2), \beta = 2m\sqrt{p^2}} \end{aligned} \quad (\text{A.14})$$

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